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Apologies for any inconvenience.

Integration by Partial Fractions

Numeracy Workshop

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Introduction

These slides are designed to review integration by the **method of partial fractions**.

Drop-in Study Sessions: Monday, Wednesday, Thursday, 10am-12pm, Meeting Room 2204, Second Floor, Social Sciences South Building, **every week.**

Website: Slides, notes, worksheets.

http://www.studysmarter.uwa.edu.au → Numeracy → Online Resources

Email: *geoff.coates@uwa.edu.au*

Next session specifically for MATH1002

Week 12: Tuesday 21/5 (12-12.45pm): Laplace transforms and the Heaviside function

The Idea

We know how to integrate polynomials such as

$$5x^2 + 2x - 4$$

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The method of partial fractions is a useful tool which allows us to integrate quotients of polynomials, such as:

$$\frac{2x^2 + 3}{x^3 - 8}$$

Important Integrals

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Example:

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So, it's easy to integrate fractions where the numerator is the derivative of the denominator.

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$$\int \frac{4}{16 + x^2}\, dx = \arctan\left(\frac{x}{4}\right) + C$$

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Sometimes, the numerator is *almost* the derivative of the denominator.

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$$2\int \frac{4x + 7}{2x^2 + 7x - 9}\,dx$$

From this, we get:

$$2\ln|2x^2 + 7x - 9| + C$$

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Now integrating is easy:

$$\frac{5}{2}\ln|2x-3| + C$$

Quotients of Linears

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$$\frac{8}{3}(3x-9).$$

If we expand this we get

$$8x-24$$

which isn't exactly equal to $8x+14$. So we adjust the constant:

$$\frac{8}{3}(3x-9)+38$$

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and separating to get

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which equals

$$\int \frac{8}{3} + \frac{38}{3}\frac{3}{3x-9}\,dx$$

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to obtain:

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Quotients of Linears

We can now easily integrate

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to obtain:

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This method of integration holds for all quotients of linears.

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If we expand this we get $2x+\frac{3}{2}$ which is not quite the same as $2x+4$. So we adjust the constant:

$$2x+4 = \frac{1}{2}(4x+3) + \frac{5}{2}$$

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So, rewriting the top gives us:

$$\int \frac{\frac{1}{2}(4x+3)+\frac{5}{2}}{2x^2+3x+5}\,dx$$

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Break this into two integrals to get:

$$\frac{1}{2}\int \frac{4x+3}{2x^2+3x+5}\,dx + \frac{5}{2}\int \frac{1}{2x^2+3x+5}\,dx$$

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The first integral is equal to:

$$\frac{1}{2}\ln\left|2x^2+3x+5\right| + C$$

The second integral is a constant over a quadratic, similiar to the arctan integral that we encountered earlier!

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This can be integrated using the arctan integral we saw before.

Partial Fractions

We now know how to integrate:

$$\frac{\text{constant}}{\text{linear}}, \quad \frac{\text{linear}}{\text{linear}}, \quad \frac{\text{constant}}{\text{quadratic}}, \quad \frac{\text{linear}}{\text{quadratic}}$$

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- For quotients where the top polynomial is of equal or higher order than the bottom, use polynomial division to convert them to a polynomial plus a remainder.
- *Any* polynomial with *real* coefficients can be factored into a product of **linear** and **quadratic** factors with real coefficients. (This is a consequence of the theorem that such polynomials are the product of linear factors with complex coefficients.)

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- A theorem from advanced algebra.

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There is a theorem which says that both linear terms on the bottom can be used to form separate fractions with a constant on top (ie. "partial fractions"):

$$\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

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Multiply through by the denominator on the LHS to get:

$$2x + 5 = A(x+2) + B(x+1)$$

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Now we substitute in convenient x values:

$$x = -2: \qquad 2(-2)+5 = A(-2+2) + B(-2+1)$$

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$$x=-2: \qquad 2(-2)+5 = A(-2+2) + B(-2+1)$$

which gives $B=-1$, and

$$x=-1: \qquad 2(-1)+5 = A(-1+2) + B(-1+1)$$

which gives $A=3$.

Partial Fractions

Therefore, the use of partial fractions has allowed us to see the equivalence:

$$\frac{2x+5}{(x+1)(x+2)} = \frac{3}{x+1} + \frac{-1}{x+2}$$

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Integrating the expression on the left is easy now; we simply integrate the two terms on the right which we know how to do!

Partial Fractions

So, each unique single linear factor gives rise to a new term.

What about if we are required to split up:

$$\frac{2x+5}{(x+1)^3(x+2)}$$

Partial Fractions

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$$\frac{2x+5}{(x+1)^3(x+2)}$$

We deal with repeated linear factors by building up as follows:

$$\frac{2x+5}{(x+1)^3(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+2)}$$

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Once again, multiply through by the denominator of the LHS and substitute in convenient x values and solve for A, B, C, D.

Partial Fractions: Irreducible Quadratics

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The x^2+1 term is irreducible (imaginary roots) and thus will not reduce into linears. We deal with this as follows:

$$\frac{2x^2+6}{(x+2)(x-1)(x^2+1)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

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Once again, convenient values of x will do the trick.

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$$\frac{2x^2+6}{(x+2)(x-1)^2(x^2+1)^3}$$

Same as repeated linears - we simply build up to the relevant power!

$$\frac{2x^2+6}{(x+2)(x-1)^2(x^2+1)^3} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} + \frac{Hx+I}{(x^2+1)^3}$$

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Once again, multiplying through by denominator of LHS and substituting in convenient values of x will do the trick.

Integration by Partial Fractions: Exercises

Evaluate the following integrals. (There are worked solutions on the following pages).

$$\textbf{Exercise 1: } \int \frac{2x+1}{x^2-11x+30}\,dx$$

$$\textbf{Exercise 2: } \int \frac{3+2x}{(4x+2)(2x+3)}\,dx$$

$$\textbf{Exercise 3: } \int \frac{x^4+2x^3+18x^2+30x+13}{(x+2)^2(x-3)}\,dx$$

Solution to Exercise 1

This one is linear over quadratic so we *could* do it by manipulating the top line to be the derivative of the bottom line. However, the bottom line factorizes so partial fractions turns out to be quicker.

$$\frac{2x+1}{x^2-11x+30} \quad = \quad \frac{2x+1}{(x-5)(x-6)}$$

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$$\begin{aligned}\frac{2x+1}{x^2-11x+30} &= \frac{2x+1}{(x-5)(x-6)} \\ &= \frac{A}{x-5}+\frac{B}{x-6}\end{aligned}$$

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Multiply through by the denominator to get:

$$2x+1 = A(x-6)+B(x-5)$$

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Multiply through by the denominator to get:

$$2x+1 = A(x-6) + B(x-5)$$

Now we substitute in convenient x values:

$$x = 5: \qquad 2(5)+1 = A(5-6) + B(5-5)$$

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$$x=5: \qquad 2(5)+1 = A(5-6)+B(5-5)$$

which gives $A=-11$, and

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$$x=6: \qquad 2(6)+1 = -11(6-6) + B(6-5)$$

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$$\frac{2x+1}{x^2-11x+30} = \frac{2x+1}{(x-5)(x-6)}$$

$$= \frac{A}{x-5} + \frac{B}{x-6}$$

Multiply through by the denominator to get:

$$2x+1 = A(x-6) + B(x-5)$$

Now we substitute in convenient x values:

$$x = 5: \qquad 2(5)+1 = A(5-6) + B(5-5)$$

which gives $A = -11$, and

$$x = 6: \qquad 2(6)+1 = -11(6-6) + B(6-5)$$

which gives $B = 13$.

Solution to Exercise 1

$$\text{Now } \int \frac{2x+1}{(x-5)(x-6)}dx \quad = \quad -11\int \frac{1}{x-5}dx + 13\int \frac{1}{x-6}dx$$

Solution to Exercise 1

$$\text{Now} \int \frac{2x+1}{(x-5)(x-6)}\,dx = -11\int \frac{1}{x-5}\,dx + 13\int \frac{1}{x-6}\,dx$$

$$= -11\ln|x-5| + 13\ln|x-6| + C$$

Solution to Exercise 2

$$\frac{3+2x}{(4x+2)(2x+3)} = \frac{A}{4x+2} + \frac{B}{2x+3}$$

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which gives $B = 0$, and

$$x = 0: \qquad 3+2(0) = A(2(0)+3)$$

which gives $A = 1$.

Solution to Exercise 2

This is an odd result! It seems that

$$\int \frac{3+2x}{(4x+2)(2x+3)}\,dx = \int \frac{1}{4x+2}\,dx.$$

In other words, we have just cancelled out the common factor of $(2x+3)$!

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If we had spotted that at the beginning, we would not have had to resort to partial fractions. (However, it's nice to know that this method arrives at this more obvious answer.)

So the solution is

$$\int \frac{1}{4x+2}\,dx = \frac{1}{4}\ln|4x+2| + C$$

Solution to Exercise 3

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$$(x+2)^2(x-3) = x^3 + x^2 - 8x - 12$$

$$x^3 + x^2 - 8x - 12 \,\big)\, \overline{x^4 + 2x^3 + 18x^2 + 30x + 13}$$

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 & x \\
\cline{2-2}
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- & x^4 + x^3 - 8x^2 - 12x \\
\cline{2-2}
 & \qquad\quad x^3 + 26x^2 + 42x + 13 \\
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\end{array}$$

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- & x^4 + \ x^3 - \ 8x^2 - 12x \\
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 & \qquad\quad x^3 + 26x^2 + 42x + 13 \\
- & \qquad\quad x^3 + \ x^2 - \ 8x - 12 \\
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- & \qquad\qquad x^3 + \;\; x^2 - \;\; 8x - 12 \\
 & \overline{\qquad\qquad\qquad 25x^2 + 50x + 25}
\end{array}$$

$$\text{Hence } \int \frac{x^4 + 2x^3 + 18x^2 + 30x + 13}{(x+2)^2(x-3)}\,dx = \int x + 1\,dx + \int \frac{25x^2 + 50x + 25}{(x+2)^2(x-3)}\,dx$$

Solution to Exercise 3

$$\frac{25x^2+50x+25}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

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$$\frac{25x^2 + 50x + 25}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

Multiply through by the denominator to get:

$$25x^2 + 50x + 25 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$$

Solution to Exercise 3

$$\frac{25x^2+50x+25}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

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Now we substitute in convenient x values:

$$x=3: \qquad 25(3)^2+50(3)+25 = A(3+2)(3-3) + B(3-3) + C(3+2)^2$$

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which gives $C = 16$,

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which gives $C = 16$,

$$x = -2: \qquad 25(-2)^2 + 50(-2) + 25 = A(-2+2)(-2-3) + B(-2-3) + 16(-2+2)^2$$

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which gives $C = 16$,

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which gives $B = -5$, and

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$$\frac{25x^2 + 50x + 25}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

Multiply through by the denominator to get:

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which gives $B = -5$, and

$$x = 0 : \qquad 25(0)^2 + 50(0) + 25 = A(0+2)(0-3) - 5(0-3) + 16(0+2)^2$$

Solution to Exercise 3

$$\frac{25x^2 + 50x + 25}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

Multiply through by the denominator to get:

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which gives $B = -5$, and

$$x = 0: \qquad 25(0)^2 + 50(0) + 25 = A(0+2)(0-3) - 5(0-3) + 16(0+2)^2$$

which gives $A = 9$.

Solution to Exercise 3

So, finally,

$$\int \frac{x^4+2x^3+18x^2+30x+13}{(x+2)^2(x-3)}$$

$$= \int x+1dx + \int \frac{25x^2+50x+25}{(x+2)^2(x-3)}dx$$

Solution to Exercise 3

So, finally,

$$\int \frac{x^4 + 2x^3 + 18x^2 + 30x + 13}{(x+2)^2(x-3)}$$

$$= \int x + 1 dx + \int \frac{25x^2 + 50x + 25}{(x+2)^2(x-3)} dx$$

$$= \int x + 1 dx + 9 \int \frac{1}{x+2} dx - 5 \int \frac{1}{(x+2)^2} dx + 16 \int \frac{1}{x-3} dx$$

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So, finally,

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$$= \frac{1}{2}x^2 + x + 9\ln|x+2| - 5 \times -\frac{1}{(x+2)} + 16\ln|x-3| + C$$

Solution to Exercise 3

So, finally,

$$\begin{aligned}
&\int \frac{x^4 + 2x^3 + 18x^2 + 30x + 13}{(x+2)^2(x-3)} \\
= \quad & \int x + 1 dx + \int \frac{25x^2 + 50x + 25}{(x+2)^2(x-3)} dx \\
= \quad & \int x + 1 dx + 9 \int \frac{1}{x+2} dx - 5 \int \frac{1}{(x+2)^2} dx + 16 \int \frac{1}{x-3} dx \\
= \quad & \frac{1}{2}x^2 + x + 9 \ln|x+2| - 5 \times -\frac{1}{(x+2)} + 16 \ln|x-3| + C \\
= \quad & \frac{1}{2}x^2 + x + 9 \ln|x+2| + \frac{5}{x+2} + 16 \ln|x-3| + C
\end{aligned}$$

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