***Partial-Fraction Decomposition:  
   General Techniques (page 1 of 3)***

*Sections: General techniques,*[*How to handle repeated and irreducible factors*](http://www.purplemath.com/modules/partfrac2.htm)*,*[*Examples*](http://www.purplemath.com/modules/partfrac3.htm)

[Previously](http://www.purplemath.com/modules/rtnladd.htm), you have added and simplified rational expressions, such as:

2/x + 1/(x + 1) = [2(x + 1)]/[x(x + 1)] + [1(x)]/[(x + 1)(x)] = ... = (3x + 2) / (x^2 + x)

Partial-fraction decomposition is the process of starting with the simplified answer and taking it back apart, of "decomposing" the final expression into its initial polynomial fractions.

To decompose a fraction, you first factor the denominator. Let's work backwards from the example above. The denominator is *x*2 + *x*, which factors as *x*(*x* + 1).

Then you write the fractions with one of the factors for each of the denominators. Of course, you don't know what the numerators are yet, so you assign variables (usually capital letters) for these unknown values:

A/x + B/(x + 1)

Then you set this sum equal to the simplified result:

(3x + 2)/[x(x + 1)] = A/x + B/(x + 1)

Multiply through by the common denominator of *x*(*x* + 1) gets rid of all of the denominators:

multiply each term by [x(x + 1)] / 1

3*x* + 2 = *A*(*x* + 1) + *B*(*x*)     Copyright © Elizabeth Stapel 2006-2011 All Rights Reserved

Multiply things out, and group the *x*-terms and the constant terms:

3*x* + 2 = *Ax* + *A*1 + *Bx* 3*x* + 2 = (*A* + *B*)*x* + (*A*)1   
(3)*x* + (2)1 = (*A* + *B*)*x* + (*A*)1

For the two sides to be equal, the coefficients of the two polynomials must be equal. So you "equate the coefficients" to get:

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3 = *A* + *B* 2 = *A*

This creates a system of equations that you can solve:

*A* = 2   
*B* = 1

Then the original fractions were (as we already know) the following:

(3x + 2) / (x^2 + x) = 2/x + 1/(x + 1)

There is another method for solving for the values of *A* and *B*. Since the equation "3*x* + 2 = *A*(*x* + 1) + *B*(*x*)" is supposed to be true for any value of *x*, we can pick useful values of *x*, plug-n-chug, and find the values for *A* and *B*. Looking at the equation "3*x* + 2 = *A*(*x* + 1) + *B*(*x*)", you can see that, if *x* = 0, then we quickly find that 2 = *A*:

3*x* + 2 = *A*(*x* + 1) + *B*(*x*)   
3(0) + 2 = *A*(0 + 1) + *B*(0)   
0 + 2 = *A*(1) + 0   
2 = *A*

And if *x* = –1, then we easily get –3 + 2 = –*B*, so *B* = 1.

I've never seen this second method in textbooks, but it can often save you a whole lot of time over the "equate the coefficients and solve the system of equations" method that they usually teach.

If the denominator of your fraction factors into unique linear factors, then the decomposition process is fairly straightforward, as shown in the previous example. But what if the factors aren't unique or aren't linear?

Sometimes a factor in the denominator occurs more than one. For instance, in the fraction13/24, the denominator 24 factors as 2×2×2×3. The factor 2 occurs three times. To get the13/24, there may have been a 1/2 or a 1/4 or a1/8 that was included in the original addition. You can't tell by looking at the final result.

In the same way, if a rational expression has a repeated factor in the denominator, you can't tell, just by looking, which denominators might have been included in the original addition. You have to account for every possibility.

* **Find the partial-fraction decomposition of the following expression:**

(x^2 + 1) / [ x (x - 1)^3 ]

The factor *x* – 1 occurs three times in the denominator. I will account for that by forming fractions containing increasing powers of this factor in the denominator, like this:

(x^2 + 1)/[x(x - 1)^3] = A/(x - 1) + B/(x - 1)^2 + C/(x - 1)^3 + D/x

Now I multiply through by the common denominator to get:

*x*2 + 1 = *Ax*(*x* – 1)2 + *Bx*(*x* – 1) + *Cx* + *D*(*x* – 1)3

I could use a system of equations to solve for *A*, *B*, *C*, and *D*, but the other method seemed easier. The two zeroing numbers are *x* = 1 and *x* = 0: so

*x* = 1: 1 + 1 = 0 + 0 + *C* + 0, so *C* = 2   
*x* = 0: 1 = 0 + 0 + 0 – *D*, so *D* = –1

But what do I do now? I have two other variables, namely *A* and *B*, for which I need values. But since I've got values for *C* and *D*, I can pick any two other *x*-values, plug them in, and get a system of equations that I can solve for *A* and *B*. The particular *x*-values I choose aren't important, so I'll pick smallish ones:

*x* = 2:   Copyright © Elizabeth Stapel 2006-2011 All Rights Reserved

(2)2 + 1 = *A*(2)(2 – 1)2 + *B*(2)(2 – 1) + (2)(2) + (–1)(2 – 1)3   
4 + 1 = 2*A* + 2*B* + 4 – 1   
5 = 2*A* + 2*B* + 3   
1 = *A* + *B*

*x* = –1:

(–1)2 + 1 = *A*(–1)(–1 – 1)2 + *B*(–1)(–1 – 1) + (2)(–1) + (–1)(–1 – 1)3   
1 + 1 = –4*A* + 2*B* – 2 + 8   
2 = –4*A* + 2*B* + 6   
2*A* – *B* = 2

I'm still stuck solving a system of equations, but by using the easier method to solve for *C* and*D*, I now have a simpler system to solve. Adding the two equations, I get 3*A* = 3, so *A* = 1. Then *B* = 0 (so that term in the expansion "vanishes"), and the complete decomposition is:

1/(x - 1) + 2/(x - 1)^3 - 1/x

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In the above example, one of the coefficients turned out to be zero. This doesn't happen often (in algebra classes, anyway), but don't be surprised if you get zero, or even fractions, for some of your coefficients. The textbooks usually stick pretty closely to nice neat whole numbers, but not always. Don't just assume that a fraction or a zero is a wrong answer. For instance:

x / (8x^2 - 10x - 3) = x / [ (4x - 3)(2x - 1) ]

...decomposes as:

(3/2) / (4x - 3) + (-1/2) / (2x - 1)

Note: You can also handle the fractions like this:

3 / [ 2(4x - 3) ] - 1 / [ 2(2x - 1) ]

If the denominator of your rational expression has an [unfactorable quadratic](http://www.purplemath.com/modules/factquad.htm), then you have to account for the possible "size" of the numerator. If the denominator contains a degree-two factor, then the numerator might not be just a number; it might be of degree one. So you would deal with a quadratic factor in the denominator by including a linear expression in the numerator.

* **Find the partial-fraction decomposition of the following:**

(x - 3) / (x^3 + 3x)

Factoring the denominator, I get *x*(*x*2 + 3). I can't factor the quadratic bit, so my expanded form will look like this:

(x - 3)/[x (x^2 + 3)] = A/x + (Bx + C)/(x^2 + 3)

Note that the numerator for the "*x*2 + 3" fraction is a linear polynomial, not just a constant term.

Multiplying through by the common denominator, I get:

*x* – 3 = *A*(*x*2 + 3) + (*Bx* + *C*)(*x*)   
*x* – 3 = *Ax*2 + 3*A* + *Bx*2 + *Cx*   
*x* – 3 = (*A* + *B*)*x*2 + (*C*)*x* + (3*A*)

The only zero in the original denominator is *x* = 0, so:

(0) – 3 = (*A* + *B*)(0)2 + *C*(0) + 3*A*   
–3 = 3*A*

Then *A* = –1. Since I have no other helpful *x*-values to work with, I think I'll take the one value I've solved for, equate the remaining coefficients, and see what that gives me:

*x* – 3 = (–1 + *B*)*x*2 + (*C*)*x* – 3   
–1 + *B* = 0  and  *C* = 1   
*B* = 1  and  *C* = 1

(There is no one "right" way to solve for the values of the coefficients. Use whichever method "feels" right to you on a given exercise.)

Then the decomposition is:

-1/x + (x + 1) / (x^2 + 3)

If the denominator of your rational expression has repeated unfactorable quadratics, then you use linear-factor numerators and follow the pattern that we used for repeated linear factors in the denominator; that is, you'll use fractions with increasing powers of the repeated factors in the denominator.

* **Set up, but do not solve, the decomposition equality for the following:**

[ x^4 + 3x - 2 ] / [ (x^2 + 1)^3 (x - 4)^2 ]

Since *x*2 + 1 is not factorable, I'll have to use numerators with linear factors. Then the decomposition set-up looks like this:

A/(x - 4) + B/(x - 4)^2 + (Cx + D)/(x^2 + 1) + (Ex + F)/(x^2 + 1)^2 + (Gx + H)/(x^2 + 1)^3

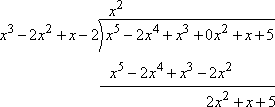
Thankfully, I don't have to try to solve this one.

One additional note: Partial-fraction decomposition only works for "proper" fractions. That is, if the denominator's degree is *not* larger than the numerator's degree (so you have, in effect, an "improper" polynomial fraction), then you first have to [use long division](http://www.purplemath.com/modules/polydiv2.htm) to get the "mixed number" form of the rational expression. Then decompose the remaining fractional part.

* **Decompose the following:**   Copyright © Elizabeth Stapel 2006-2011 All Rights Reserved

(x^5 - 2x^4 + x^3 + x + 5) / (x^3 - 2x^2 + x - 2)

The numerator is of degree 5; the denominator is of degree 3. So first I have to do the long division:



The long division rearranges the rational expression to give me:

x^2 + (2x^2 + x + 5) / (x^3 - 2x^2 + x - 2)

Now I can decompose the fractional part. The denominator factors as (*x*2 + 1)(*x* – 2).

x^2 + (2x^2 + x + 5) / [ (x^2 + 1)(x - 2) ]

The *x*2 + 1 is irreducible, so the decomposition will be of the form:

(2x^2 + x + 5)/(x^3 - 2x^2 + x - 2) = A/(x - 2) + (Bx + C)/(x^2 + 1)

Multiplying out and solving, I get:

2*x*2 + *x* + 5 = *A*(*x*2 + 1) + (*Bx* + *C*)(*x* – 2)   
*x* = 2: 8 + 2 + 5 = *A*(5) + (2*B* + *C*)(0), 15 = 5*A*, and *A* = 3   
*x* = 0: 0 + 0 + 5 = 3(1) + (0 + *C*)(0 – 2),   
          5 = 3 – 2*C*, 2 = –2*C*, and *C* = –1   
*x* = 1: 2 + 1 + 5 = 3(1 + 1) + (1*B* – 1)(1 – 2),   
          8 = 6 + (*B* – 1)(–1) = 6 – *B* + 1,   
          8 = 7 – *B*, 1 = – *B*, and *B* = –1

Then the complete expansion is:

x^2 + 3/(x - 2) + (-x - 1)/(x^2 + 1) = x^2 + 3/(x - 2) - (x + 1)/(x^2 + 1)

The preferred placement of the "minus" signs, either "inside" the fraction or "in front", may vary from text to text. Just don't leave a "minus" sign hanging loose underneath.