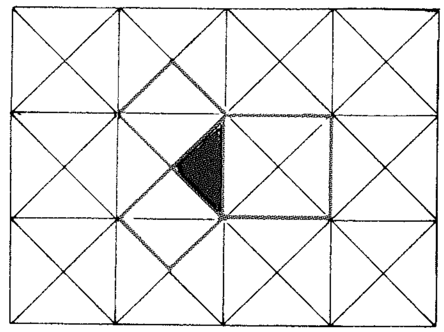


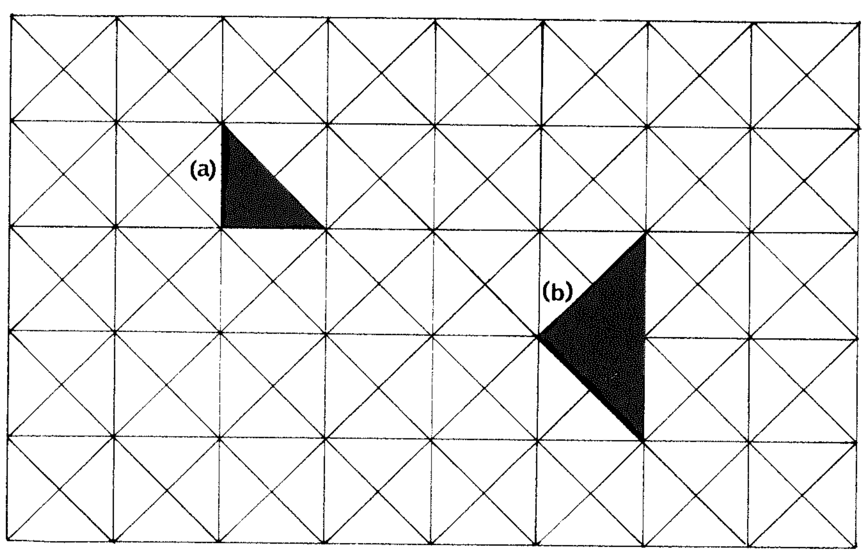
The above diagram shows a pattern for a tiled floor. Each tile is a right isosceles triangle.

(a) How many tiles are needed to make the large square?

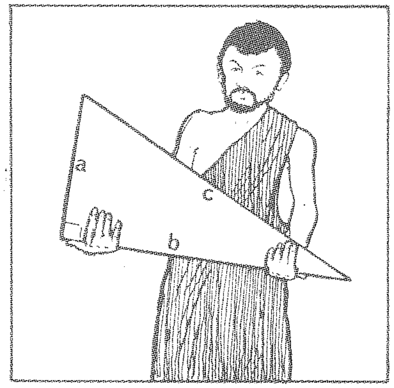
(b) How many tiles are needed altogether to make the other two squares?

(c) Is the number of tiles needed for the large square equal to the number needed to make the other two squares?

4. Check your results of the previous question for each of the triangles shown below.



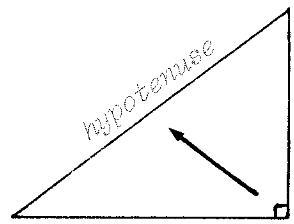
SECTION 2(a)
THE PYTHAGOREAN THEOREM



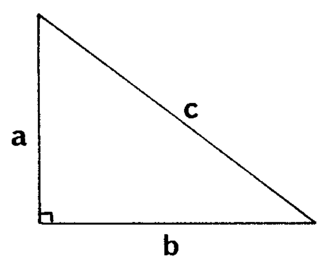
The activities of the previous section have introduced you to the Pythagorean Theorem. The results of these activities may be summarized by the following statement of the Pythagorean Theorem:

'In any right triangle, the area of the square drawn on the longest side of the triangle is equal to the sum of the areas of the squares drawn on each of the two shorter sides.'

NOTE: The longest side in a right triangle is opposite the right angle and is called the *hypotenuse*.

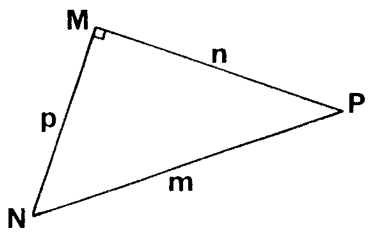


This statement of the theorem is generally shortened to:

> In any right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.
>
> 
>
>
> $$c^2 = a^2 + b^2$$

EXERCISE 2(a)

1.



(a) Name the hypotenuse of this right triangle.

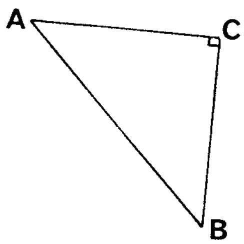
(b) Which of the following statements is true?

$m^2 = n^2 + p^2$

$n^2 = m^2 + p^2$

$p^2 = m^2 + n^2$

2.



(a) Which side is the hypotenuse of this right triangle?

(b) Which of the following statements is true?

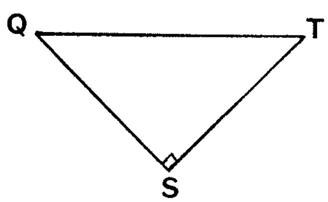
$AC^2 = AB^2 + BC^2$

$AB^2 = AC^2 + BC^2$

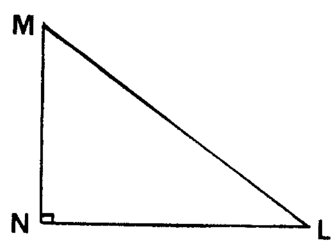
$BC^2 = AB^2 + AC^2$

3. For each triangle drawn below write the appropriate Pythagorean relationship.

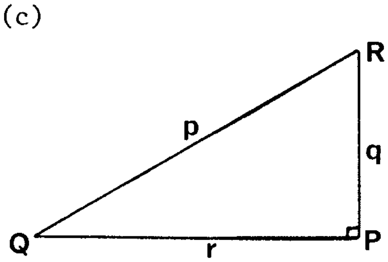
(a)

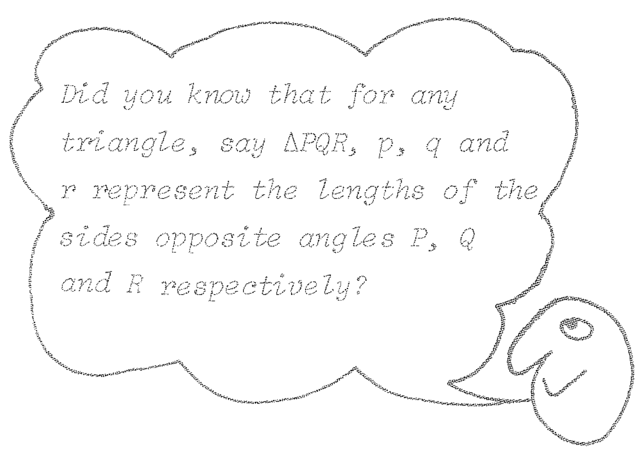


(b)

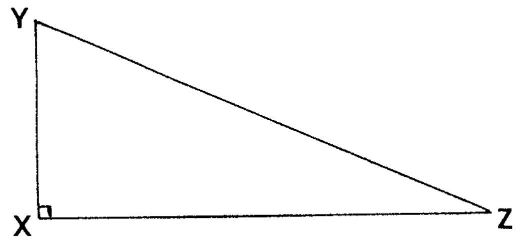


(c)





4.



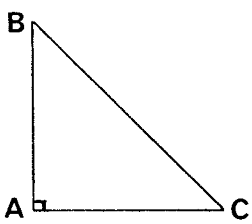
The Pythagorean relationship for this triangle can be written a number of ways all equivalent to one another.

$YZ^2 = XY^2 + XZ^2$ $\qquad$ $x^2 = z^2 + y^2$

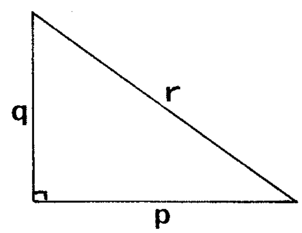
$XY^2 + XZ^2 = YZ^2$ $\qquad$ $z^2 + y^2 = x^2$

Can you see how to write this same statement in other ways?

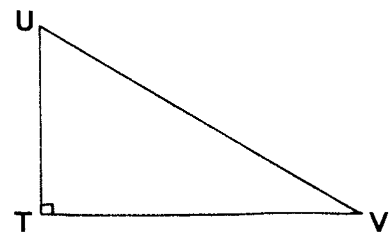
5. Complete each of the following statements of the Pythagorean theorem.



(a) $AC^2 =$

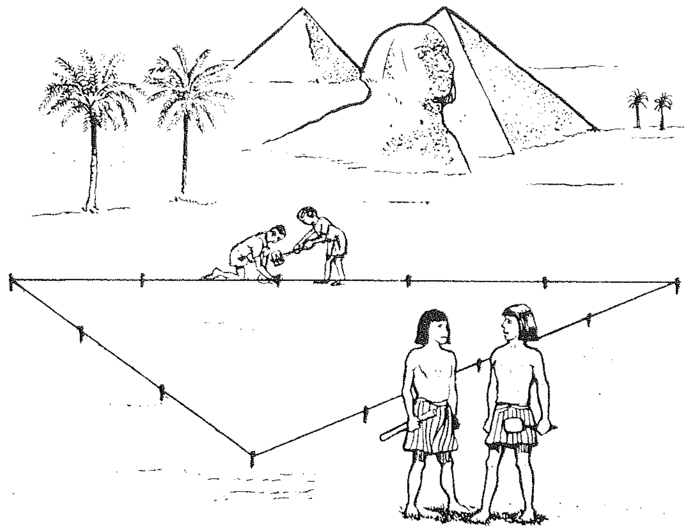


(b) $q^2 =$



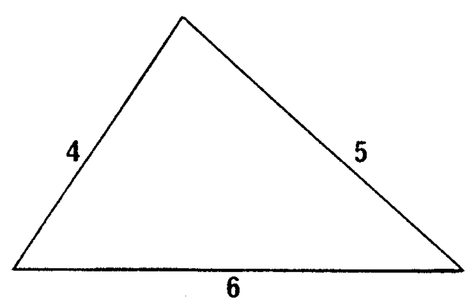
(c) $u^2 =$

SECTION 2(b) PYTHAGOREAN TRIPLES



The ancient Egyptians stumbled across the fact that if a triangle of sides 3, 4 and 5 units is taken, the largest angle turns out to be a right angle (see page 7). Is this a coincidence, or will triangles with sides of 4, 5 and 6 units; 5, 6 and 7 units; 6, 7 and 8 units and so on, also be right angled?

Look at the *4, 5, 6 triangle* drawn below.



The area of the square on the longest side is:

6^2 or 36 square units.

The sum of the areas of squares on the other two sides is:

$4^2 + 5^2$ or $16 + 25$ or 41 square units.

In this case, does the square on the longest side equal the sum of the squares on the other two sides?

Measure the angle sizes of this triangle. Is the *4, 5, 6 triangle* a right triangle?

You should have answered 'no' to each of the above questions.

Let us explore this situation a little further.

ACTIVITY:

For each triangle ABC in the table below, draw the triangle accurately and fill in the required information.

	Sides of ΔABC in cm			a^2	b^2	$a^2 + b^2$	c^2	Is $a^2 + b^2 = c^2$?	Is ΔABC a right triangle?
	a	b	c						
1.	4	5	6	16	25	41	36	No	No
2.	4	5	7						
3.	4	5	8						
4.	5	6	7						
5.	5	6	8						
6.	5	12	13						

The results tabled above illustrate the *converse* of the Pythagorean Theorem. It states:

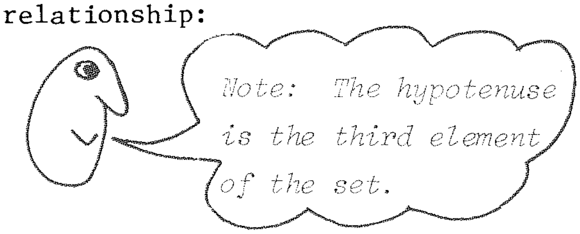
> 'If the sum of the squares of the two shorter sides of a triangle is equal to the square on the third side, then the triangle is a right triangle.'

We can use this result to test for right triangles given the length of the three sides.

If three counting numbers a, b and c satisfy the relationship:

$$a^2 + b^2 = c^2$$

then {a, b, c} is called a *Pythagorean triple*.



Can you suggest why?

EXERCISE 2(b)

1. Which of the following sets are Pythagorean triples?

   (a) {6, 8, 10} (d) {12, 16, 20} hypotenuse

   (b) {6, 9, 11} (e) {12, 16, 24}

   (c) {9, 12, 15} (f) { 5, 12, 13}

2. Examine the following sets of Pythagorean triples.

   {3, 4, 5}

   {6, 8,10}

   {9,12,15}

   (a) Do you notice any relationship between these sets?

   (b) Can you decide whether {15,20,25} is a Pythagorean triple *without* finding the square of each of the elements? Explain your answer.

   (c) For n a counting number, is {3n, 4n, 5n} a Pythagorean triple?

   Your answers to these questions should have led you to discover that *the multiples of Pythagorean triples are also Pythagorean triples*.

3. Find the value of x in the following to make Pythagorean triples:

   (a) {10, 24, x} (d) { x, 40, 50} hypotenuse

   (b) { 9, x, 15} (e) {10, x, 26}

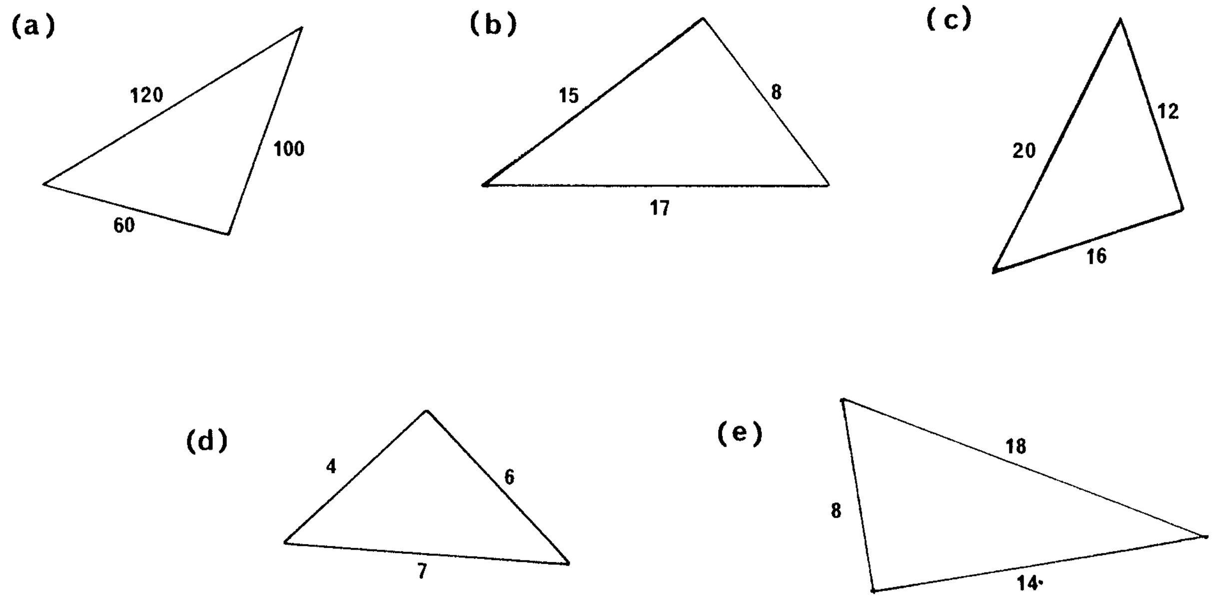
   (c) { x, 5, 13} (f) { 8, 15, x}

4. By examining your results of the activity on page 16, complete the table below:

	Sides of ΔABC in cm a	b	c	a^2	b^2	$a^2 + b^2$	c^2	Is $a^2 + b^2 > c^2$ $a^2 + b^2 = c^2$ $a^2 + b^2 < c^2$	Is ΔABC acute right obtuse?
1.	4	5	6	16	25				
2.	4	5	7	16	25				
3.	4	5	8	16	25				
4.	5	6	7	25	36				
5.	5	6	8	25	36				
6.	5	12	13	25	144				

Check your answers before proceeding to the next question.

5. Now, using the method developed in question 4, classify each of the following triangles as right, acute or obtuse. (Do not use a protractor.)



FOR INTEREST

6. Pythagoras and his followers developed the following rule for generating Pythagorean triples.

'If m is any *odd* whole number greater than 1, then m, $\frac{1}{2}(m^2 - 1)$ and $\frac{1}{2}(m^2 + 1)$ will be a Pythagorean triple.'

(a) Find the Pythagorean triple generated when m = 5.

(b) Find the Pythagorean triple generated when m = 7.

(c) Why are Pythagorean triples *not* generated when m is even?

*(d) A triangle has sides of length n, n + 1 and n + 2 units (where n is a counting number).
Show that there is only one value of n for which the triangle is right angled.
What is this value of n?

7. The Greeks had also developed another procedure for generating Pythagorean triples:

'Pythagorean triples $\{a, b, c\}$ will be given by $a = 2uv$, $b = u^2 - v^2$ and $c = u^2 + v^2$ where $u > v$ and u and v are counting numbers.'

Use this rule to generate Pythagorean triples for:

(a) $u = 2, \quad v = 1$

(b) $u = 4, \quad v = 3$

(c) $u = 4, \quad v = 1$

8. Repeat the procedure used in question 7 for:

(a) $u = 3, \quad v = 1$

(b) $u = 4, \quad v = 2$

(c) $u = 6, \quad v = 3$

Now look carefully at the triples generated in question 7 and those generated in question 8.

9. Can you see any difference in types of triples between those generated in question 7 and question 8?

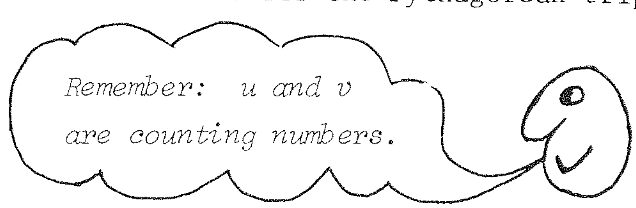
Those in question 7 have no common factor and are called *primitive* triples. Those in question 8 have a common factor and are called *composite* triples.

We find that primitive triples are generated when:

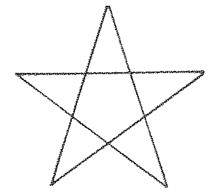
(i) u and v have no common factor and

(ii) u and v are of different parity (one odd and one even).

*10. If $u > v$, the expressions $2uv$, $u^2 - v^2$ and $u^2 + v^2$ may be used to find Pythagorean triples. Find the values of u and v for the Pythagorean triple $\{48, 55, 73\}$.



SECTION 3 THE SECRET OF THE PYTHAGOREAN BROTHERHOOD



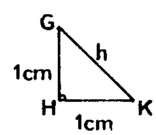
The pentagram - secret badge of the Pythagorean Brotherhood.

It is no challenge to ask any of you to draw a line segment of length 2 cm, but can you draw one of length $\sqrt{2}$ cm or $\sqrt{5}$ cm?

As you will discover in this section, the Pythagoreans developed a method for drawing line segments of length $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ units and so on, but were puzzled because they still could not attach a real *value* to these numbers.

(The story surrounding the Pythagorean Brotherhood's problem will be told later in this section.)

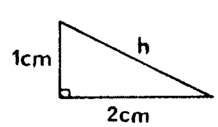
A length of $\sqrt{2}$ cm can be drawn by considering a right isosceles triangle with sides each of length 1 cm.



The Pythagorean Theorem states:

$$\begin{aligned} GK^2 &= GH^2 + HK^2 \\ \text{i.e.}\quad h^2 &= (1\text{cm})^2 + (1\text{cm})^2 \\ \text{i.e.}\quad h^2 &= 1\text{cm}^2 + 1\text{cm}^2 \\ \text{i.e.}\quad h^2 &= 2\text{cm}^2 \\ \text{i.e.}\quad h &= \sqrt{2}\ \text{cm} \end{aligned}$$

Now try a right triangle of sides 1 cm and 2 cm.



What is the length of the hypotenuse of this triangle?

ANSWERS

EXERCISE 1 page 10

1. (a) Squares A, B and D.

   (b) Length of side of square A is 25 mm; area of square A is 625 mm^2.
   Length of side of square B is 20 mm; area of square B is 400 mm^2.
   Length of side of square D is 15 mm; area of square D is 225 mm^2.
   Hence area of square A = area of square B + area of square D.

2. (a), (c)

3. (a) 4

   (b) 4

   (c) Yes.

4. (a) 8 tiles for the large square; 4 tiles for each of the smaller squares.

   (b) 16 tiles for the large square; 8 tiles for each of the smaller squares.
   The same result holds as for question 3.

EXERCISE 2(a) page 13

1. (a) $\overline{NP}$

   (b) $m^2 = n^2 + p^2$

2. (a) $\overline{AB}$

   (b) $AB^2 = AC^2 + BC^2$

3. (a) $QT^2 = QS^2 + ST^2$

   (b) $ML^2 = MN^2 + NL^2$

   (c) $p^2 = q^2 + r^2$ or $QR^2 = RP^2 + PQ^2$

4. $XY^2 = YZ^2 - XZ^2$, $z^2 = x^2 - y^2$
   $XZ^2 = YZ^2 - XY^2$ $y^2 = x^2 - z^2$

5. (a) $AC^2 = BC^2 - AB^2$

   (b) $q^2 = r^2 - p^2$

   (c) $u^2 = t^2 - v^2$

EXERCISE 2(b) page 17

1. (a), (c), (d), (f).

2. (a) Each set contains multiples of the numbers 3, 4 and 5.
   (b) Yes, {15, 20, 25} is a set of multiples of 3, 4 and 5.
   (c) Yes, {3n, 4n, 5n} is a Pythagorean triple.

3. (a) 26 (d) 30
   (b) 12 (e) 24
   (c) 12 (f) 17

4.

	Sides of ΔABC in cm a	b	c	a^2	b^2	$a^2 + b^2$	c^2	Is $a^2 + b^2 > c^2$ $a^2 + b^2 = c^2$ $a^2 + b^2 = c^2$?	Is ABC acute right obtuse?
1.	4	5	6	16	25	41	36	$a^2 + b^2 > c^2$	acute
2.	4	5	7	16	25	41	49	$a^2 + b^2 < c^2$	obtuse
3.	4	5	8	16	25	41	64	$a^2 + b^2 < c^2$	obtuse
4.	5	6	7	25	36	61	49	$a^2 + b^2 > c^2$	acute
5.	5	6	8	25	36	61	64	$a^2 + b^2 < c^2$	obtuse
6.	5	12	13	25	144	169	169	$a^2 + b^2 = c^2$	right

5. (a) obtuse
   (b) right
   (c) right
   (d) acute
   (e) obtuse

6. (a) {5, 12, 13}
   (b) {7, 24, 25}
   (c) When m is even, m^2 is even and m^2-1 and m^2+1 are odd. Hence the values of $\frac{1}{2}(m^2-1)$ and $\frac{1}{2}(m^2+1)$ will not be counting numbers, so $\{m, \frac{1}{2}(m^2-1), \frac{1}{2}(m^2+1)\}$ will not be a Pythagorean triple.

(d) $n + 2$ will be the longest side in the right triangle.

By the Pythagorean Theorem,

if $(n + 2)^2 = (n + 1)^2 + n^2$

then $n^2 + 4n + 4 = n^2 + 2n + 1 + n^2$

i.e. $n^2 + 4n + 4 = 2n^2 + 2n + 1$

i.e. $n^2 - 2n - 3 = 0$

i.e. $(n - 3)(n + 1) = 0$ — *If a.b = 0 then a=0 or b=0*

which means either $n - 3 = 0$ or $n + 1 = 0$

i.e. $n = 3$ $n = -1$

But if n is a counting number, $n = 3$.

7. (a) $u = 2$, $v = 1$ $a = 4$, $b = 3$, $c = 5$. The triple is {3, 4, 5}
   (b) $u = 4$, $v = 3$ $a = 24$, $b = 7$, $c = 25$. The triple is {7, 24, 25}
   (c) $u = 4$, $v = 1$ $a = 8$, $b = 15$, $c = 17$. The triple is {8, 15, 17}

8. (a) $u = 3$, $v = 1$ $a = 6$, $b = 8$, $c = 10$. The triple is {6, 8, 10}
   (b) $u = 4$, $v = 2$ $a = 16$, $b = 12$, $c = 20$. The triple is {12, 16, 20}
   (c) $u = 6$, $v = 3$ $a = 36$, $b = 27$, $c = 45$. The triple is {27, 36, 45}

10. $u = 8$, $v = 3$

EXERCISE 3 page 24

1. (a) (d) (e) (g) (h)

2. (a) 2
   (b) 0·6
   (c) 7·5
   (d) $\sqrt{2}$ or 1·41 correct to two decimal places.
   (e) 5
   (f) $\sqrt{27}$ or $3\sqrt{3}$ or 5·20 correct to two decimal places.

3. (a) 5·66
   (b) 0·12
   (c) 52·92

4. QT = 15, RT = 12 then ST = 12·5

EXERCISE 4(a) page 36

1. 10 m

2. 8 mm

3. 25 m

4. 13 km

5. 1·8 m

6 (a) 10·91
(b) 15
(c) 6·24
(d) 420
(e) 8·25

7. $20\sqrt{2}$ or 28·28 m correct to two decimal places

8. $40\sqrt{2}$ or 56·57 mm correct to two decimal places.

9. $25\sqrt{3}$ or 43·30 mm^2 correct to two decimal places.

10. 13 m

11. $6\sqrt{3}$ or 10·39 units

12. (a) $\sqrt{5 + 2\sqrt{2}}$ or 2·80 km correct to two decimal places.
(b) $(\sqrt{5 + 2\sqrt{2}} - \sqrt{5})$ or 0·56 km shorter (correct to two decimal places).

PUZZLE page 36

12 cubits

FOR INTEREST page 37

$A_2 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2 = \frac{1}{2}(9\pi)$ square units

$A_3 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4)^2 = \frac{1}{2}(16\pi)$ square units