CORNERSTONE CHRISTIAN COLLEGE

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YEAR 12 3AB Mathematics Semester 2

Investigation 2

Number Patterns:
Square Numbers Pythagorean Triples and the Fibonacci Sequence

Reading Material

This material is designed to remind you of some of the basic number patterns you have studied previously and to extend some of the ideas beyond what you have encountered in the classroom.

The investigation will be based on the three areas of square numbers, Pythagorean triples and the Fibonacci sequence. It will involve algebraic and numerical exploration of the patterns, but will not be strictly limited to the material or relationships explored here.

Square numbers

From Wikipedia, the free encyclopedia: http://en.wikipedia.org/wiki/Square_number

In mathematics, a **square number** or **perfect square** is an integer that is the square of an integer;[1] in other words, it is the product of some integer with itself. For example, 9 is a square number, since it can be written as 3×3.

The usual notation for the formula for the square of a number n is not the product $n \times n$, but the equivalent exponentiation n^2, usually pronounced as "n squared". The name *square* number comes from the name of the shape; see below.

Square numbers are non-negative. Another way of saying that a (non-negative) number is a square number, is that its square roots are again integers. For example, $\sqrt{9} = \pm 3$, so 9 is a square number.

A positive integer that has no perfect square divisors except 1 is called square-free.

For a non-negative integer n, the nth square number is n^2, with $0^2 = 0$ being the 0-th one. The concept of square can be extended to some other number systems. If rational numbers are included, then a square is the ratio of two square integers, and, conversely, the ratio of two square integers is a square (e.g., $4/9 = (2/3)^2$).

Contents

Examples

The squares (sequence A000290 in OEIS) smaller than 60^2 are:

$0^2 = 0$
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$
$11^2 = 121$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$16^2 = 256$
$17^2 = 289$
$18^2 = 324$
$19^2 = 361$
$20^2 = 400$
$21^2 = 441$
$22^2 = 484$
$23^2 = 529$
$24^2 = 576$
$25^2 = 625$
$26^2 = 676$
$27^2 = 729$
$28^2 = 784$
$29^2 = 841$
$30^2 = 900$
$31^2 = 961$
$32^2 = 1024$
$33^2 = 1089$
$34^2 = 1156$
$35^2 = 1225$
$36^2 = 1296$
$37^2 = 1369$
$38^2 = 1444$
$39^2 = 1521$
$40^2 = 1600$
$41^2 = 1681$
$42^2 = 1764$
$43^2 = 1849$
$44^2 = 1936$
$45^2 = 2025$
$46^2 = 2116$
$47^2 = 2209$
$48^2 = 2304$
$49^2 = 2401$
$50^2 = 2500$
$51^2 = 2601$
$52^2 = 2704$
$53^2 = 2809$
$54^2 = 2916$
$55^2 = 3025$
$56^2 = 3136$
$57^2 = 3249$
$58^2 = 3364$
$59^2 = 3481$

The difference between any perfect square and its predecessor is given by the identity

$$n^2 - (n-1)^2 = 2n - 1.$$

Equivalently, it is possible to count up square numbers by adding together the last square, the last square's root, and the current root, that is, $n^2 = (n-1)^2 + (n-1) + n$.

Properties

The number m is a square number if and only if one can compose a square of m equal (lesser) squares:

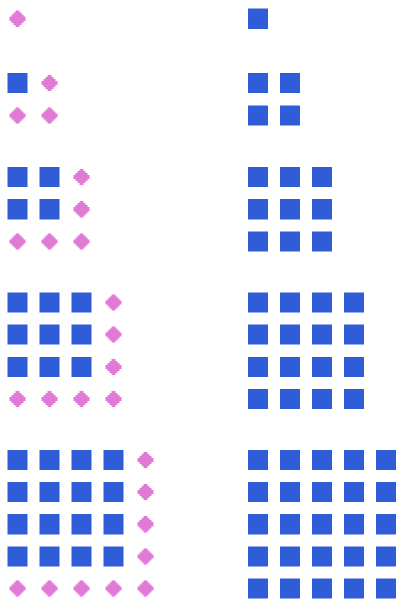
$m = 1^2 = 1$

$m = 2^2 = 4$

$m = 3^2 = 9$

$m = 4^2 = 16$

$m = 5^2 = 25$



Note: White gaps between squares serve only to improve visual perception.
There must be no gaps between actual squares.

The unit of area is defined as the area of unit square (1×1). Hence, a square with side length n has area n^2.

The expression for the nth square number is n^2. This is also equal to the sum of the first n odd numbers as can be seen in the above pictures, where a square results from the previous one by adding an odd number of points (shown in magenta). The formula follows:

$$n^2 = \sum_{k=1}^{n} (2k-1).$$

So for example, $5^2 = 25 = 1 + 3 + 5 + 7 + 9$.

There are several recursive methods for computing square numbers. For example, the nth square number can be computed from the previous square by

$n^2 = (n-1)^2 + (n-1) + n = (n-1)^2 + (2n-1)$. Alternatively, the nth square number can be calculated from the previous two by doubling the $(n − 1)$-th square, subtracting the $(n − 2)$-th square number, and adding 2, because $n^2 = 2(n − 1)^2 − (n − 2)^2 + 2$. For example,

$$2 \times 5^2 - 4^2 + 2 = 2 \times 25 - 16 + 2 = 50 - 16 + 2 = 36 = 6^2.$$

A square number is also the sum of two consecutive triangular numbers. The sum of two consecutive square numbers is a centered square number. Every odd square is also a centered octagonal number.

Another property of a square number is that it has an odd number of positive divisors, while other natural numbers have an even number of positive divisors. An integer root is the only divisor that pairs up with itself to yield the square number, while other divisors come in pairs.

Lagrange's four-square theorem states that any positive integer can be written as the sum of four or fewer perfect squares. Three squares are not sufficient for numbers of the form $4^k(8m + 7)$. A positive integer can be represented as a sum of two squares precisely if its prime factorization contains no odd powers of primes of the form $4k + 3$. This is generalized by Waring's problem.

A square number can end only with digits 0, 1, 4, 6, 9, or 25 in base 10, as follows:

1. If the last digit of a number is 0, its square ends in an even number of 0s (so at least 00) and the digits preceding the ending 0s must also form a square.
2. If the last digit of a number is 1 or 9, its square ends in 1 and the number formed by its preceding digits must be divisible by four.
3. If the last digit of a number is 2 or 8, its square ends in 4 and the preceding digit must be even.
4. If the last digit of a number is 3 or 7, its square ends in 9 and the number formed by its preceding digits must be divisible by four.
5. If the last digit of a number is 4 or 6, its square ends in 6 and the preceding digit must be **odd**.
6. If the last digit of a number is 5, its square ends in 25 and the preceding digits must be 0, 2, 06, or 56.

In base 16, a square number can end only with 0, 1, 4 or 9 and

- in case 0, only 0, 1, 4, 9 can precede it,
- in case 4, only even numbers can precede it.

In general, if a prime p divides a square number m then the square of p must also divide m; if p fails to divide m/p, then m is definitely not square. Repeating the divisions of the previous sentence, one concludes that every prime must divide a given perfect square an even number of times (including possibly 0 times). Thus, the number m is a square number if and only if, in its canonical representation, all exponents are even.

Squarity testing can be used as alternative way in factorization of large numbers. Instead of testing for divisibility, test for squarity: for given m and some number k, if $k^2 - m$ is the square of an integer n then $k - n$ divides m. (This is an application of the factorization of a difference of two squares.) For example, $100^2 - 9991$ is the square of 3, so consequently $100 - 3$ divides 9991. This test is deterministic for odd divisors in the range from $k - n$ to $k + n$ where k covers some range of natural numbers $k \geq \sqrt{\boxed{m}}$.

A square number cannot be a perfect number.

The sum of the series of power numbers

$$\sum_{n=0}^{N} n^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + N^2$$

can also be represented by the formula

$$\frac{N(N+1)(2N+1)}{6}.$$

The first terms of this series (the square pyramidal numbers) are:

0, 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, 4900, 5525, 6201... (sequence A000330 in OEIS).

All fourth powers, sixth powers, eighth powers and so on are perfect squares.

Special cases

- If the number is of the form *m*5 where *m* represents the preceding digits, its square is *n*25 where $n = m \times (m + 1)$ and represents digits before 25. For example the square of 65 can be calculated by $n = 6 \times (6 + 1) = 42$ which makes the square equal to 4225.
- If the number is of the form *m*0 where *m* represents the preceding digits, its square is *n*00 where $n = m^2$. For example the square of 70 is 4900.
- If the number has two digits and is of the form 5*m* where *m* represents the units digit, its square is *AABB* where $AA = 25 + m$ and $BB = m^2$. Example: To calculate the square of 57, 25 + 7 = 32 and $7^2 = 49$, which means $57^2 = 3249$.
- If the number ends in 5, its square will end in 5; similarly for ending in 25, 625, 0625, 90625, ... 8212890625, etc. If the number ends in 6, its square will end in 6, similarly for ending in 76, 376, 9376, 09376, ... 1787109376. For example, the square of 55376 is 3066501376, both ending in *376*. (The numbers 5, 6, 25, 76, etc. are called automorphic numbers. They are sequence A003226 in the OEIS.)

Odd and even square numbers

Squares of even numbers are even (and in fact divisible by 4), since $(2n)^2 = 4n^2$.

Squares of odd numbers are odd, since $(2n + 1)^2 = 4(n^2 + n) + 1$.

It follows that square roots of even square numbers are even, and square roots of odd square numbers are odd.

As all even square numbers are divisible by 4, the even numbers of the form $4n + 2$ are not square numbers.

As all odd square numbers are of the form $4n + 1$, the odd numbers of the form $4n + 3$ are not square numbers.

Squares of odd numbers are of the form $8n + 1$, since $(2n + 1)^2 = 4n(n + 1) + 1$ and $n(n + 1)$ is an even number.

Every odd perfect square is a centered octagonal number. The difference between any two odd perfect squares is a multiple of 8. The difference between 1 and any higher odd perfect square always is eight times a triangular number, while the difference between 9 and any higher odd perfect square is eight times a triangular number minus eight. Since all triangular numbers have an odd factor, but no two values of 2^n differ by an amount containing an odd factor, the only perfect square of the form $2^n - 1$ is 1, and the only perfect square of the form $2^n + 1$ is 9.

Pythagorean Triples Reading

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Pythag/pythag.html#section3

3.1 Hypotenuse and Longest side are consecutive

3, 4, 5
5, 12, 13
7, 24, 25
9, 40, 41
11, 60, 61

The first and simplest Pythagorean triangle is the 3, 4, 5 triangle. Also near the top of the list is 5, 12, 13. In both of these *the longest side and the hypotenuse are consecutive integers*.
The list here shows there are more.
Can you spot the pattern?

Can we find a formula for these triples?

You will have noticed that the *smallest sides are the odd numbers 3, 5, 7, 9,...* So the smallest sides are of the form $2i+1$.
The other sides, as a series are 4, 12, 24, 40, 60... Can we find a formula here?
We notice they are all multiples of 4: 4×1, 4×3, 4×6, 4×10, 4×15,.. . The series of multiples: 1, 3, 6, 10, 15,... are the Triangle Numbers with a formula $^{i(i+1)}/_{2}$.
So our second sides are 4 times each of these, or, simply, just $2i(i+1)$.
The third side is just one more than the second side: $2i(i+1)+1$, so our formula is as follows:

shortest side = $2i+1$; longest side = $2i(i+1)$; hypotenuse = $1+2i(i+1)$

Check now that the sum of the squares of the two sides is the same as the square of the hypotenuse (Pythagoras's Theorem).

i	a:2i+1	b:2i(i+1)	h=b+1
1	3	2×1×2=4	5
2	5	2×2×3=12	13
3	7	2×3×4=24	25
4	9	2×4×5=40	41
5	11	2×5×6=60	61
6	13	2×6×7=84	85
7	15	2×7×8=112	113

Bill Batchelor points out that the sum of the two consecutive sides is $4i^2 + 4i + 1$ which is just the square of the smallest side.
This gives us an alternative method of generating these triples:

- Take an odd number as the smallest side: e.g. 9
- square it (81) - which will be another odd number
- split the square into two halves (40·5), rounding one down (40) and the other rounded up (41), to form the other two sides (9, 40, 41)

Alternatively, let's look at the m,n values for each of these triples. Since the hypotenuse is one more than a leg, the 3 sides have no common factor so are primitive and therefore all of them do have m,n values:

Triple	m	n
3, 4, 5	2	1
5, 12, 13	3	2
7, 24, 25	4	3
9, 40, 41	5	4
11, 60, 61	6	5

It is easy to see that $m = n + 1$.
The m,n formula in this case gives
$a = m^2 - n^2 = (n+1)^2 - n^2 = 2n + 1$
$b = 2mn = 2(n+1)n = 2n^2 + 2n$
$h = m^2 + n^2 = (n+1)^2 + n^2 = 2n^2 + 2n + 1$
So h is $b+1$ and the pattern is always true:

if $m = n+1$ in the m,n formula then it generates a (primitive) triple with hypotenuse = 1 + the longest leg.

Are these all there are? Perhaps there are other m,n values with a leg and hypotenuse consecutive numbers. In fact, they are all given by the formula above because:
The triangles must be primitive so we know they have an m,n form.
$h = m^2 + n^2$ could be either one more than either $m^2 - n^2$ or $2mn$:

> If $h = m^2 + n^2 = \mathbf{(m^2 - n^2)} + 1$
> then, taking m^2 from both sides: $n^2 = -n^2 + 1$
> $2n^2 = 1$ or $n^2 = 1/2$ and this is not possible for a whole number n
> So h is never $a + 1$.
> If $h = m^2 + n^2 = \mathbf{2mn} + 1$ then
> $m^2 - 2mn + n^2 = 1$ which is the same as
> $(m - n)^2 = 1$. Therefore
> $m - n = 1$ or $m - n = -1$
> But the hypotenuse is a positive number and is $m^2 - n^2$ so we must have so $m>n$ and so $m - n$ cannot be -1

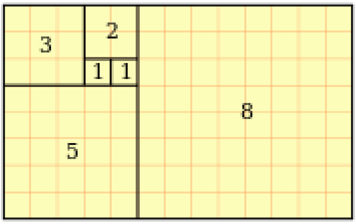
The only condition we can have is that $m - n = 1$ or $m = n + 1$ - there are no other triangles with hypotenuse one more than a leg except those generated by consecutive m n values in the m,n formula.

See Also "The Pythagorean Theorem" pp15-19, 74,75

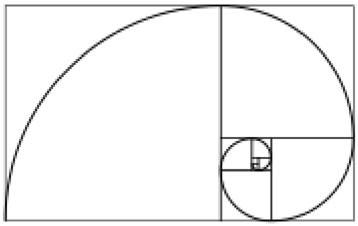
Fibonnaci Sequence

Fibonacci number

From Wikipedia, the free encyclopedia http://en.wikipedia.org/wiki/Fibonacci_number



A tiling with squares whose side lengths are successive Fibonacci numbers



An approximation of the golden spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling; this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34.

In mathematics, the **Fibonacci numbers** or **Fibonacci sequence** are the numbers in the following integer sequence:[1][2]

$$1,\;1,\;2,\;3,\;5,\;8,\;13,\;21,\;34,\;55,\;89,\;144,\;\ldots$$

or (often, in modern usage):

$$0,\;1,\;1,\;2,\;3,\;5,\;8,\;13,\;21,\;34,\;55,\;89,\;144,\;\ldots$$ (sequence A000045 in OEIS).

By definition, the first two numbers in the Fibonacci sequence are 1 and 1, or 0 and 1, depending on the chosen starting point of the sequence, and each subsequent number is the sum of the previous two.

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with seed values[1][2]

$$F_1 = 1,\; F_2 = 1$$

or[3]

$$F_0 = 0,\; F_1 = 1.$$

The Fibonacci sequence is named after Fibonacci. His 1202 book *Liber Abaci* introduced the sequence to Western European mathematics,[4] although the sequence had been described earlier in Indian mathematics.[5][6][7] By modern convention, the sequence begins either with $F_0 = 0$ or with $F_1 = 1$. The *Liber Abaci* began the sequence with $F_1 = 1$, without an initial 0.

Fibonacci numbers are closely related to Lucas numbers in that they are a complementary pair of Lucas sequences. They are intimately connected with the golden ratio; for example, the closest rational approximations to the ratio are 2/1, 3/2, 5/3, 8/5, ... Applications include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings,[8] such as branching in trees, phyllotaxis (the arrangement of leaves on a stem), the fruit sprouts of a pineapple,[9] the flowering of an artichoke, an uncurling fern and the arrangement of a pine cone.[10]

Origins

A page of Fibonacci's *Liber Abaci* from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Latin and Roman numerals and the value in Hindu-Arabic numerals.

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody.[6][11] In the Sanskrit oral tradition, there was much emphasis on how long (L) syllables mix with the short (S), and counting the different patterns of L and S within a given fixed length results in the Fibonacci numbers; the number of patterns that are *m* short syllables long is the Fibonacci number F_{m+1}.[7]

Susantha Goonatilake writes that the development of the Fibonacci sequence "is attributed in part to Pingala (200 BC), later being associated with Virahanka (c. 700 AD), Gopāla (c. 1135), and Hemachandra (c. 1150)".[5] Parmanand Singh cites Pingala's cryptic formula *misrau cha* ("the two are mixed") and cites scholars who interpret it in context as saying that the cases for *m* beats (F_{m+1}) is obtained by adding a [S] to F_m cases and [L] to the F_{m-1} cases. He dates Pingala before 450 BC.[12]

However, the clearest exposition of the series arises in the work of Virahanka (c. 700 AD), whose own work is lost, but is available in a quotation by Gopala (c. 1135):

> Variations of two earlier meters [is the variation]... For example, for [a meter of length] four, variations of meters of two [and] three being mixed, five happens. [works out examples 8, 13, 21]... In this way, the process should be followed in all *mātrā-vṛttas* [prosodic combinations].[13]

The series is also discussed by Gopala (before 1135 AD) and by the Jain scholar Hemachandra (c. 1150).

In the West, the Fibonacci sequence first appears in the book *Liber Abaci* (1202) by Leonardo of Pisa, known as Fibonacci.[4] Fibonacci considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?

- At the end of the first month, they mate, but there is still only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

At the end of the *n*th month, the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month $n - 2$) plus the number of pairs alive last month ($n - 1$). This is the *n*th Fibonacci number.[14]

The name "Fibonacci sequence" was first used by the 19th-century number theorist Édouard Lucas.[15]

List of Fibonacci numbers

The first 21 Fibonacci numbers F_n for $n = 0, 1, 2, ..., 20$ are:[16]

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765